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A Method for Constructing Non-Isosceles Triangular Fuzzy Numbers Using Frequency Histogram and Statistical Parameters

A. Amini[1*], N. Nikraz[2]

1. Ph.D. Student, Faculty of Science and Engineering, Curtin University, Kent St, Bentley WA 6102, Australia
2. Senior Lecturer, Faculty of Science and Engineering, Curtin University, Kent St, Bentley WA 6102, Australia

Corresponding author: *amin.amini@postgrad.curtin.edu.au*

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ABSTRACT

The philosophy of fuzzy logic was formed by introducing the membership degree of a linguistic value or variable instead of divalent membership of 0 or 1. Membership degree is obtained by mapping the variable on the graphical shape of fuzzy numbers. Because of simplicity and convenience, triangular membership numbers (TFN) are widely used in different kinds of fuzzy analysis problems. This paper suggests a simple method using statistical data and frequency chart for constructing non-isosceles TFN when we are using direct rating for evaluating a variable in a predefined scale. In this method, the relevancy between assessment uncertainties and statistical parameters such as mean value and the standard deviation is established in a way that presents an exclusive form of triangle number for each set of data. The proposed method with regard to the graphical shape of the frequency chart distributes the standard deviation around the mean value and forms the TFN with the membership degree of 1 for mean value. In the last section of the paper modification of the proposed method is presented through a practical case study.

1. Introduction

One of the most important steps in solving the problems and analyzing the systems by using fuzzy logic is defining the fuzzy membership functions of the data set. Fuzzy control systems, fuzzy inference engines, fuzzy multi-criteria decision-making models and ranking system based on fuzzy logic use fuzzy membership functions as input. So a more accurate defined membership

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function results in a more accurate output and higher efficiency of fuzzy analysis systems. This paper proposed a novel and simple method for constructing the triangular membership function using frequency chart of a certain set of statistical data when the average point of data is considered the most possible value. This set can be the collected information from a survey using linguistic judgments or qualitative assessments expressed in a numerically defined scale to find an answer to the question: " How *F* is *a*?" where '*F'* is a fuzzy concept, and '*a'* is a parameter which is being assessed. In different sections of this paper, after a short review of basic fuzzy logic concepts and membership function construction methods, we introduced the proposed method through some numerical examples.

2. Fuzzy and classic logic

In the classical logic, a simple proposition '*P*' is a linguistic, or declarative statement contained within a universe of elements, X, that can be identified as being a collection of elements in X that are strictly true or strictly false [1]. In classical logic, a binary truth value is assigned to the veracity of an element in the proposition '*P'*, which is a value of 1 (truth) or 0 (false). For example, consider the '*P'* statement as: "water with the temperature over 60 centigrade degree is hot", based on classical logic, water to 59.9 degrees is not considered hot water at all. So there is a crisp boundary between true and false in classical logic, which causes making decisions about processes that contain nonrandom uncertainty, such as the uncertainty in natural language, be less than perfect. Treating truth as a linguistic variable leads to a fuzzy linguistic logic, or merely fuzzy logic [2]. The original fuzzy logic founded by Lotfi Zadeh as a key to decision-making when faced with linguistic and non-random uncertainty. Fuzzy logic is a precise logic of imprecision and approximate reasoning [3]. It may be viewed as an attempt at formalization/mechanization of two remarkable human capabilities; First, the capability to converse, reason and make rational decisions in an environment of imprecision, uncertainty, incompleteness of information, conflicting information, partiality of truth and partiality of possibility - in short, in an environment of imperfect information- and second, the capability to perform a wide variety of physical and mental tasks without any measurements and any computation [4].

In Fuzzy logic, a statement can be either true or false and also can be neither true nor false. Fuzzy logic is non-monotonic logic. It is a superset of conventional logic that has been extended to handle the concept of partial truth, the truth values between 'completely true' and 'completely false'. It is a type of logic that recognizes more than simple true and false values. With fuzzy logic, propositions can be represented with degrees of truthfulness and falsehood. For example, the statement "today is sunny" might be 100% true if there are no clouds, 80% true if there are a few clouds, 50% true if it is hazy and 0% true if it rains all day.

3. Fuzzy set vs. Crisp set

In contrast to classical set theory, each element either fully belongs to the set or is completely excluded from the set. In other words, classical set theory represents a special case of the more general fuzzy set theory. In the crisp set, membership of element X, $\mu_A(X)$ of set A is defined as:

$$\mu_A(X) = \begin{cases} 1 & X \in A \\ 0 & X \notin A \end{cases} \quad (1)$$

For example, figure 1.a, shows a crisp set of height between 5 to 7 feet, thus every height in this range has the same value of truth equals to 1 which means it belongs to this set, and every height out of this range has a value of 0 that means this value doesn't belong to this set.

Dr. Zadeh developed the concept of 'fuzzy sets' to account for numerous concepts used in human reasoning which are vague and imprecise, e.g. tall, old [5]. In his paper of 1965 he stated: "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.''

A fuzzy set expresses the degree to which an element belongs to a set. If X is a collection of objects denoted generically by x, then a fuzzy set A in X is defined as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) \mid x \in X\} \quad , \quad \mu_A(X) \in [0,1] \quad (2)$$

The characteristic function of a fuzzy set, μ_A (x) is allowed to have values between 0 and 1, which denotes the degree of membership of an element in a given set and is called as 'membership function' (or MF for short) If the values of the membership function is restricted to either 0 or 1, then A is reduced to a classical set [6]. In figure 1.b, a fuzzy set of heights between 5 and 7 feet and around 6 has been illustrated. In this example the fuzzy set A may be described as follows: A = {(5, 0), (5.5, 0.5), (6, 1), (6.5, 0.5), (7, 0)}.

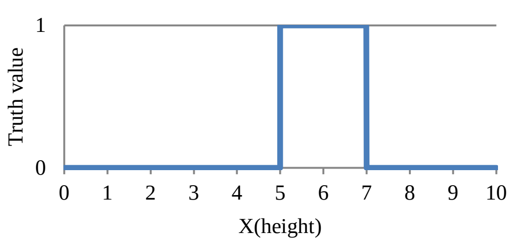


Fig. 1. a) A crisp set of height between 5 to 7 feet.

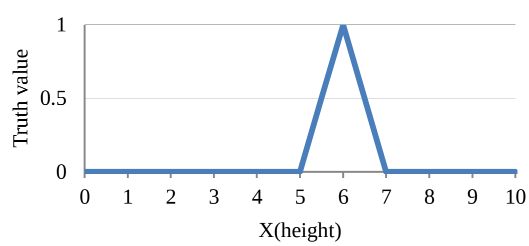


Fig. 1 .b) A fuzzy set of height around 6 feet.

Fuzzy sets are often incorrectly assumed to indicate some form of probability. Even though they can take on similar values, it is important to realize that membership grades are not probabilities. The probabilities of a finite universal set must add to 1 while there is no such requirement for membership grades.

In this paper, we use the partial truth concept in the form of fuzzy membership function to show the truth degree of the average point of a set of data collected based on a scaled assessment system.

4. Fuzzy set vs. the fuzzy number

A fuzzy number is a fuzzy set on the real numbers. It represents information such as 'about m'. A fuzzy number must have a unique modal value 'm', be convex, normal and piecewise continuous [7]. Fuzzy numbers generalize real classical numbers and roughly speaking a fuzzy number is a fuzzy subset of the real line that has some additional properties. They are capable of modeling epistemic uncertainty and its propagation through calculations. The fuzzy number concept is basic for fuzzy analysis and fuzzy differential equations, and a very useful tool in several applications of fuzzy sets and fuzzy logic [8].

A fuzzy set is not a fuzzy number since it is not fuzzy convex and normal. An alternative and more direct definition of convexity is the following [5]: A is convex if and only if for all x_1 and x_2 in X and all λ in [0, 1]:

$$f_A[\lambda x_1 + (1-\lambda)x_2] \geq Min[f_A(x_1), f_A(x_2)] \quad (3)$$

A fuzzy set A is normal if we can always find a point x ∈ X such that $\mu_A(X) = 1$. The shape (a) represented in figure 2, is a fuzzy set, not a fuzzy number and shape (b) in that figure is a convex set, but not a normal one.

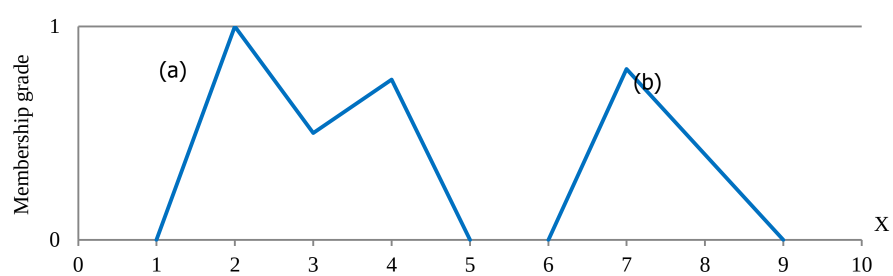


Fig. 2. Normal but not convex fuzzy set (a) and convex but not normal fuzzy set (b).

A fuzzy set is completely characterized by its membership function (MF). A membership function associated with a given fuzzy set maps an input value to its appropriate membership value. The only condition a membership function must satisfy to be considered a fuzzy number is that it must vary between 0 and 1. The function itself can be an arbitrary curve whose shape we can define as a function that suits us from the point of view of simplicity, convenience, speed, and efficiency.

4.1. L-R fuzzy numbers

There are various types of membership functions, e.g., S-shaped function, Z-shaped function, triangular membership function, trapezoidal membership function, Gaussian distribution function, an exponential function, Pi function and vicinity function [8]. A more convenient and concise way to define an MF is to express it as a mathematical formula. Dubois and Prade [9], introduced the concept of L-R approximations of fuzzy numbers and replaced the convolution type operations by interval based ones. All of the mentioned membership functions are presentable in the form of L-R fuzzy numbers. An L-R fuzzy number (or interval) u has the membership function of the form [10]:

$$\mu_u(x) = \begin{cases} f_L\left(\frac{x-a}{b-a}\right) & if\ x \in [a,b] \\ 1 & if\ x \in [b,c] \\ f_R\left(\frac{d-x}{d-c}\right) & if\ x \in [c,d] \\ 0 & otherwise \end{cases} \tag{4}$$

Where f_L, f_R: [0, 1] → [0, 1] are two continuous, increasing functions, fulfilling $f_L(0)=f_R(0)=0$, $f_L(1)=f_R(1)=1$. The compact interval [a, d] is the support and the core is [b, c]. The usual notation is u = (a, b, c, d), f_L, f_R for an interval and u = (a, b, c) for a number.

L-R fuzzy numbers are considered important in the theory of fuzzy sets and their particular cases as triangular and trapezoidal fuzzy numbers, when the functions f_L and f_R are linear, are very useful in applications. These straight line membership functions have the advantage of simplicity.

The trapezoidal membership function has a flat top and is just a truncated triangle curve. A 'trapezoidal MF' is specified by four parameters {a, b, c, d} as follows: (a ≤ b ≤ c ≤ d) [11]. Figure 3, illustrates a trapezoidal MF defined by trapezoid (x; 1, 3, 6, 9).

$$\text{Trapezoid}(x; a, b, c, d) = \begin{cases} 0 & x \le a \\ \frac{(x-a)}{(b-a)} & a \le x \le b \\ 1 & b \le x \le c \\ \frac{(d-x)}{(d-c)} & c \le x \le d \\ 0 & d \le x \end{cases} \tag{5}$$

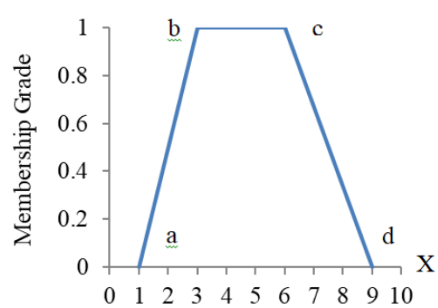


Fig. 3. A trapezoidal MF.

4.2. Triangular membership function

The simplest MF is the triangular membership function. A triangular MF is specified by three parameters {a, b, c} as follows:

$$\text{Triangular}\,(x;a,b,c)=\begin{cases}0 & x\le a\\ \frac{(x-a)}{(b-a)} & a\le x\le b\\ \frac{(c-x)}{(c-b)} & b\le x\le c\\ 0 & c\le x\end{cases} \tag{6}$$

The parameters {a, b, c} (with a < b< c) determine the x coordinates of the three corners of the underlying triangular MF. Figure 4, illustrates a triangular MF defined by a triangle (x; 1, 3, 7) on a 10-grade scale which can be based on ten fuzzy linguistic values or 10 pre-defined conditions such as effectiveness grade, importance degree, agreement level, etc.

A fuzzy uncertain quantity has a range of values between the lowest possible limit (below which there are no possible values) and highest possible limit (beyond which there are no possible values). The membership grades represent the degrees of belief in the truth levels of the values in the range of the fuzzy number. The three corners of a TFN present the lowest possible value (a), the possible value (b), and the highest possible value (c). The values in the range between the lowest and highest possible values have a membership grade between 0 and 1, with the possible value having a membership grade of 1. The lowest and highest possible values have membership grades of 0 because they represent the lower and upper limits of the fuzzy range outside which no values belong to the fuzzy number. The membership grade for a given value in the range between the lowest possible value and the highest possible value is evaluated using linear interpolation by finding the membership grade on the straight line corresponding to a given value in the fuzzy range.

5. Membership value assignments

By summarizing subjective versus objective on one dimension and individual versus group, on the other hand, Biligic and Turksen [12] considered five categories of interpretations for production of membership functions. They discussed these interpretations for the meaning of $\mu_T(x) = 0.7$, represented for the vague expression: "John (x) is tall (T)", where $\mu_T(x)$ is the membership degree of *x* defined on a fuzzy set tall (T), as:

1. Likelihood view: 70% of a given population consider John as a tall person.
2. Random set view: 70% of a given population described 'tall' as an interval containing John's height.
3. Similarity view (typicality view): to the degree 0.3 (a normalized distance), John's height is away from the prototypical object, which is truly "tall".
4. Utility view: the utility of confidence that John is tall is 0.7.
5. Measurement view: when compared to others, John is taller than some, and this privilege is 0.7.

After introducing eight methods: polling, direct rating (point estimation), reverse rating, interval estimation (set-valued statistics), membership function exemplification, clustering methods and neural-fuzzy methods for constructing the membership function in their paper (See [13] for details and original references) they discussed measurement theory [14] as a framework which can find the appropriate method for each type of interpretation. Where in the direct rating [12] the parameter or variable is being classified according to a fuzzy concept (like importance degree, tallness, darkness,…) and the question is: "How *F* is *a*?", in polling technique we find the membership functions values proportional to positive answers to a presented subject. The question in this method is: "Do you consider *a* as *F*?" where '*a*' is the parameter, and '*F*' is a fuzzy concept. In such kind of indirect way, we can define an interval scale and generate the membership value based on the frequencies that each interval gets when the scale is being questioned by a group of experts. In other words, each interval gets a weight equal to the number of agreement [15].

In [16] Saaty proposed a pairwise comparison matrix for computing the membership values. The entries of this matrix were relative preference defined on a rational scale. Introducing the possibility theory against the probability theory by Zadeh [17] opened a new vision for many authors to study the conversion problem of probability distribution to possibility distribution when membership functions are considered numerically equivalent to possibility distribution. Two famous transformation methods are bijective transformation by Dubois and Prade [18] and the conservation of the uncertainty method by Klir [19].

In [20] Civanlar and Trussell proposed a membership function generation method for statistically based data. They believed that the membership function has a relationship to some physical property of the set, so they considered two properties for membership functions derived from statistics: making some allowance for deviation from the value obtained by the measurement and being naturally quantitative. The produced membership functions using their method are optimal with respect to a set of reasonable criteria and also adjustable to possibility-probability consistency principle.

Valliappan and Pham [21] discussed a membership function construction method using subjective and objective information. The subjective part is experts' opinions, and judgments and the objective part are statistical dates and their known probability density function (pdf). In the proposed framework assumptions of the "program-evaluation and review technique" (PERT) was used to derive the normalized subjective measures through the beta distribution. Then, by using the kernel of the fuzzification, the subjective part is transformed into a fuzzy set.

In [22] Chen and Otto suggested a method using measurement theory and constrained interpolation for constructing the membership function in a way that they used a measurement scale construction for a given finite set of determined membership values and determined the remaining membership values using interpolation. Witold Pedrycz [23] has shown that the routinely used triangular membership functions provide an immediate solution to the optimization problems emerging in fuzzy modeling.

Whereas describing all the methods and efforts done in constructing the membership functions is beyond the scope of this paper, most famous methods and techniques have been summarized in Table 1. The major part of this table is based on studies done by Medasani et al. [24], Sancho-

Royo and Verdegay [25] and Sivanandam and Sumathi [26] about different methods and techniques for membership functions generation.

Many of these techniques are not applicable to many practical problems involved prevailing uncertainty or in multi-attribute decision-making problems where we need to have convex and normal fuzzy numbers as input weights to form the decision-making matrix. However, the technique proposed in this paper is categorized as a subjective/direct rating method and use the frequency histogram of a parameter which has been evaluated by a group of experts on a graded scale; it tries to utilize the objective data in a way to emphasize the principle of uncertainty and imprecise judgment and generate unique triangular fuzzy numbers. In a numerical example, the discussed method is compared to other subjective methods of polling and direct rating for a better understanding of the differences.

Table 1
Different methods and techniques for membership functions construction.

Membership Function Generating Methods	Applied Techniques
Subjective perception based methods	▪ Interval estimation ▪ Continues direct valuation ▪ Direct rating ▪ Reverse rating ▪ Polling ▪ Pairwise comparison(Relative preference) ▪ Parameterized MF(Based on distance from ideal state or deductive reasoning)
Heuristic methods	▪ Piecewise linear functions(linearly increasing, linearly decreasing or a combination of these) ▪ Piecewise monotonic functions(S-functions, Sin(x), π-Functions, exponential functions,…)
Histogram-based methods	▪ Modeling multidimensional histogram using a combination of parameterized functions
Transformation of probability distributions to possibility distributions	▪ Bijective transformation method ▪ Conservation of uncertainty method
Fuzzy nearest neighbor method	▪ K-nearest neighbors(K-NN)
Neural network based methods	▪ Feed forward multilayer neural networks
Clustering based methods	▪ Fuzzy C-Means(FCM) ▪ Robust agglomerative Gaussian mixture decomposition(RAGMD) ▪ Self-Organizing feature map(SOFM)
Genetic Algorithm	▪ Fitness function evaluation
Inductive Reasoning	▪ Entropy minimization(Clustering the parameters corresponding to the output classes)

6. The proposed method for constructing a non-isosceles triangular fuzzy number

In recent decades using fuzzy theory in management and engineering has increased significantly. Fuzzy science is able to construct models which can process qualitative information intelligently almost like a human.

The first step of every fuzzy analysis is fuzzification. Fuzzification [26] is the process of converting a real scalar value into a fuzzy value. This is achieved with the different types of fuzzifiers or membership functions. In a multi-criteria decision-making problem, decision matrix entries and weight vectors are fuzzy rather crisp numbers. Fuzzy ranking problems, the items or options introduced in the form of fuzzy numbers are being prioritized using different fuzzy ranking methods. In fuzzy management, knowledge, and skills needed to manage the systems can be obtained from experts in natural language and create models and computer programs easily by using fuzzy inference engines. In this case, natural language often uses the attributes and constraints, such as ‘very’, ‘little’, ‘some’ and ‘approximately’ that can be shown by membership functions and give as input to computer programs [27]. As mentioned before, fuzzy triangular numbers are very useful in all kinds of problems using fuzzy theory because of simplicity and ease.

Now we need to answer this question that “What’s new with our proposed method?” Ordinary methods which use statistical data to generate fuzzy triangular numbers use the normal distribution of data for this purpose. The normal distribution is a continuous probability distribution that shows the probability that any real observation will fall between any two real limits or real numbers (Fig. 5.a). The ordinary method of converting a normal distribution function to a TFN results in an isosceles triangular fuzzy number. (Fig. 5.b.)

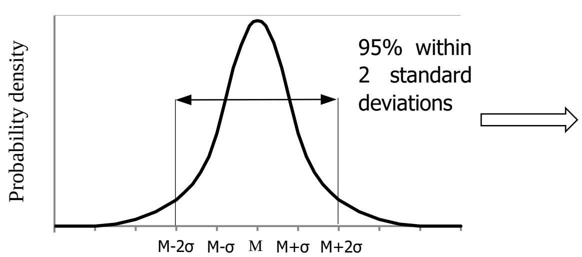


Fig. 5. a) Normal distribution function with mean value of *M* and standard deviation of *σ*

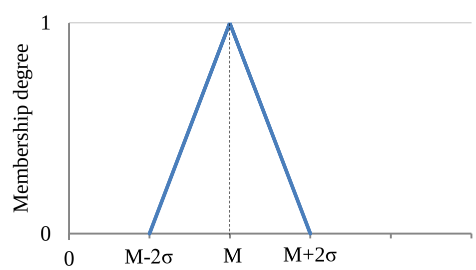


Fig. 5. b) An isosceles TFN with an interval of 2 standard deviations

This paper suggests a simple method for constructing non-isosceles triangular fuzzy number (TFN) of an item, parameter, value or concept which has been surveyed statistically via questionnaire, interview or other investigating methods based on utility view for constructing the membership degree using a pre-defined scale for converting the linguistic judgments or cluster distances to quantitative values. In another word the proposed method converts the data of a frequency chart to corresponding TFN. The origin or main idea for generating fuzzy membership function by this method is the deviation of the responses from the average value when a fuzzy concept is judged or rated. Applying this method in a practical case study will be discussed later for the verification.

The triangular fuzzy number could not be in the form of a simple isosceles triangle with two equal sides when the statistical data distribution around the mean point is not homogenous. Thus, for constructing the TFN that can represent the judgment deviations, we try to determine the left and right boundaries about the average value of data by proposing the following steps:

Computing the average or mean value of frequency chart data using equation (7) that is presented by point ‘M’ and standard deviation of data ‘σ’using equation (8).

$$M = \frac{1}{n}\sum_{i=1}^{n}(x_i) \tag{7}$$

$$\sigma = \sqrt[2]{\frac{1}{n}\sum_{i=1}^{n}(x_i - M)^2} \tag{8}$$

Forming the histogram of the frequency chart in the form of a continuous graph which introduced by *f*(x), where the *X*-axis indicates the scale degrees and Y-axis indicates the frequency data.

Introducing and computing the following parameters with regard to *f*(x) for a '*0* to *k'* graded X-axis:

$$L_M = \int_0^M f(x)dx \tag{9}$$

$$R_M = \int_M^k f(x)dx \tag{10}$$

$$S = \frac{L_M}{R_M} \tag{11}$$

$$\sigma_R\,(M) = \frac{\sigma}{(1+S)} \tag{12}$$

$$\sigma_L\,(M) = \frac{\sigma.s}{(1+s)} \tag{13}$$

Finding the lower limit *(LL)* and the upper limit *(UL)*: So that the lower limit is obtained by subtracting the $\sigma_L(M)$ from the mean value and the upper limit is obtained by adding the $\sigma_R(M)$ to the mean value. In the presentation of a TFN in equation (6), *(LL)* and *(UL)* are point *'a'* and point *'c'*, respectively.

$$(LL) = M - \sigma_L(M) \tag{14}$$

$$(UL) = M + \sigma_R(M) \tag{15}$$

Scaling the data in the form of fuzzy number membership function with a membership degree of *1* for the mean point and the membership degree of zero for the lower limit *(LL)* and upper limit *(UL)*.

$$triangular\,(x; LL, M, UL) = \begin{cases} 0 & x \le LL \\ \frac{(x-LL)}{(M-LL)} & LL \le x \le M \\ \frac{(UL-x)}{(UL-M)} & M \le x \le UL \\ 0 & UL \le x \end{cases} \tag{16}$$

Referred to the equation (9) and (10), 'L_M' is the area under the frequency graph for the left side of the mean point and 'R_M' is the area under this diagram on the right side of the mean point. $\sigma_L(M)$ and $\sigma_R(M)$ are left and right boundaries of the fuzzy number. These values are obtained by distributing the standard deviation value (σ) of data regarding the ratio *'S'* using the direct proportion. In this way, the side with bigger area due to more scattered responses leads to a bigger boundary around the average value, which represents less certainty and more vagueness.

6.1. Numerical example 1:

Through a numerical example, we try to show the described method steps clearer. Table 2 shows the frequency chart of a rated parameter evaluated by 80 experts in a 10-grade scale that can be the importance degree or weight or degree of impact of a parameter, so the question is: "*How important is parameter i (P_i)* ?" Values from 0 to 10 can define 10 different fuzzy states or linguistic expressions. Where the score (0) indicates the unimportance of being, (1) too little importance, (2) the relatively low importance, (3) low importance, (4) the low average, (5) the average, (6) the upper average, (7) the relatively high, (8) high importance, (9) very high importance and (10) is a special importance [28].

In the conversion of statistical data into fuzzy numbers, Continuous fuzzy numbers are used. Thus, the distances between these 10 points become meaningful.

Table 2
Frequency chart for the importance degree of a surveyed parameter.

Rating scale	0	1	2	3	4	5	6	7	8	9	10	Total
Frequency of Responses	2	4	5	10	24	16	5	8	3	1	2	80

If the rating scale represents the importance degree of the parameter, data of table 2 shows that 5 experts realized that this parameter has the importance degree 2 (the relatively low importance) whereas 2 out of 80 inquired experts considered a 10 importance degree (special importance) for this parameter. The corresponding diagram for the parameter is obtained from the frequency chart. The graph represents the frequency of values for the sample parameter has been illustrated in figure 6. The sum of total frequencies is equal to the number of the inquired experts, which are 80 people.

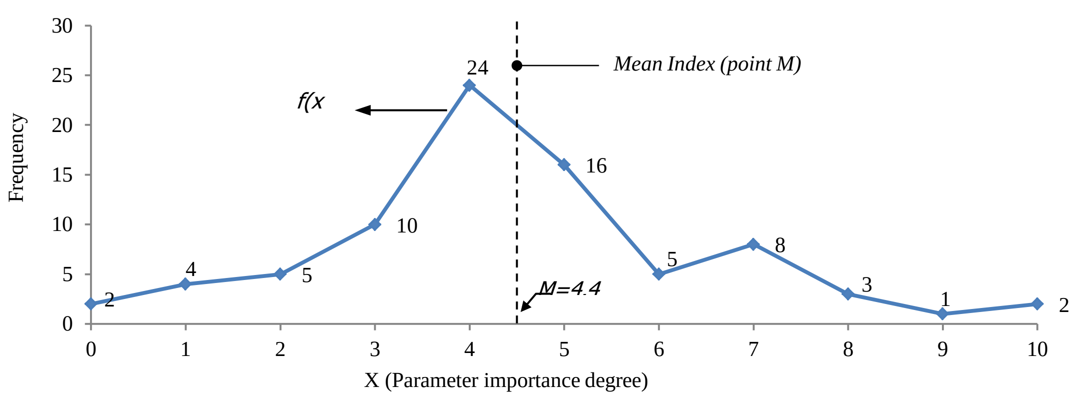


Fig. 6. Continuous diagram of frequency for parameter importance degree based on a ten grade scale.

The result of the proposed algorithm for obtaining the membership function for the sample parameter has been shown in table 3. The related membership function has been illustrated in figure 7.

Table 3
The results of the proposed algorithm for calculating the lower and upper limits of the sample parameter.

Parameter	Mean (M)	Standard deviation (σ)	$S=(L_a/R_a)$	$\sigma_R(M)$	$\sigma_L(M)$	Lower limit (LL)	Upper limit (UL)
Value	4.49	2.04	1.38	0.86	1.188	3.30	5.35

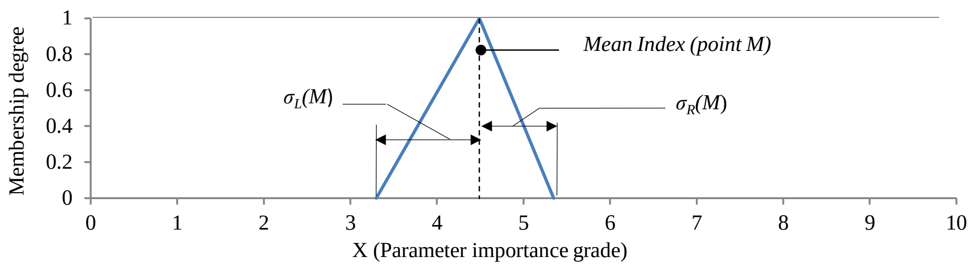


Fig. 7. Corresponding TFN of parameter importance grade based on a ten grade evaluating scale.

This method reflects the uncertainty of qualitative judgments because it is exclusive for each set of evaluation. For example, two sets of data with the same average value will not have the same TFN because of different standard deviations and also the different distribution of frequency histogram around the mean point. In this method, a smaller standard deviation indicated a more certain set of assessment or judgment that results in narrower boundaries of the fuzzy number about the average point. For example, in a case that all the experts consider a parameter with an average degree of importance equals to 5, the standard deviation would be zero (0), so there isn't any boundary around the single point core, and the fuzzy triangle turn into a fuzzy singleton number. A singleton fuzzy number shows that there isn't any doubt or uncertainty about the importance degree of the parameter (Fig. 8).

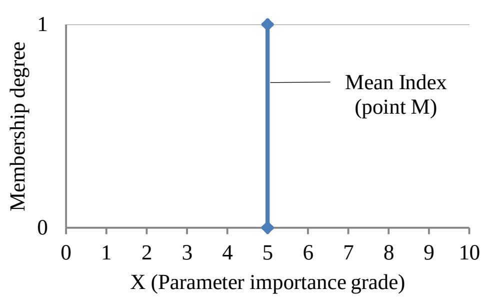


Fig. 8. Fuzzy singleton in 100 percent certainty state

6.2. Numerical example 2:

Consider the fuzzy subject '*F*' is 'warmth' and the variable *x* is 50° water. We try to find the membership degree of 50° water via asking the opinions of 40 people through some kinds of subjective methods and proposed technique for a better understanding of the differences between them.

6.2.1. Polling method

The responses to this question: “is 50° water warm?” with ‘*yes*’ or ‘*no*’ have been presented in table 4. If we calculate the positive answers to this question proportional to all responses, the membership degree for “warmth” of 50° would be 0.875. Repeating this question for a range of temperatures may lead to a membership function for warmth illustrated in figure 9.

Table 4
Polling frequency chart for warmth of 50° water.

“Is 50° water warm?”	**Yes**	**No**	**Membership degree of ‘*Yes*’**
Frequency	35	5	**(35/40) = 0.875**
Total	**40**		

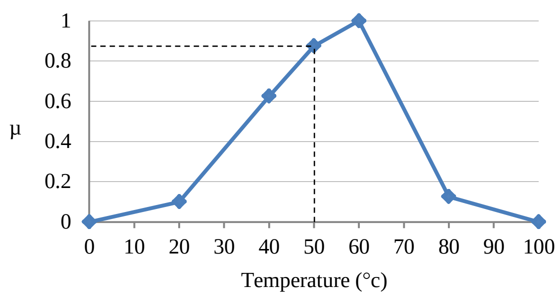


Fig. 9. MF of warmth for different temperature degrees using polling method.

6.2.2. Direct rating

we can assign a number from 1 to 10 to “*How 50° water is warm*?” and rate the degree of warmth of 50° water. We reach to a fuzzy function for “How 50° water is warm” by assigning the membership degree of *1* for the maximum frequency and finding the other grades proportional to the largest frequency these results are summarized in table 5.

Table 5
Frequency chart for a sample surveyed parameter.

	“How is 50° water warm?”										
Rating scale	0	1	2	3	4	5	6	7	8	9	**10**
Frequency	0	0	0	0	0	0	0	0	5	15	**20**
Membership degree	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0.25**	**0.75**	**1**

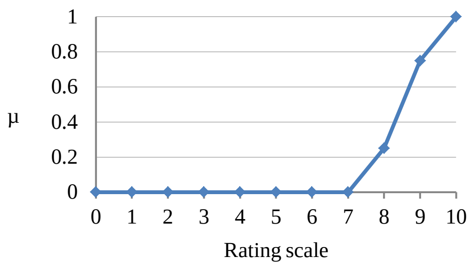


Fig. 10. b) MF of "how 50° water is warm".

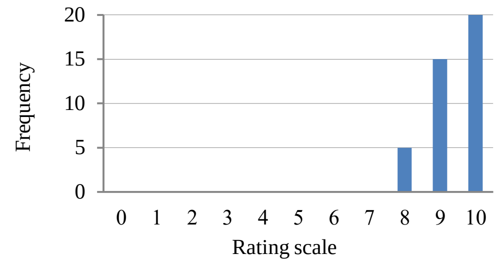


Fig. 10. a) Frequency histogram of direct rating of 50° water' warmth.

So how should we use direct rating for forming the diagram of a fuzzy concept (warmth in this example) for a range of variables (different temperature degrees) when many experts are being asked to express their opinion? Turksen and Norwich in [29] described a method for constructing the diagram for a linguistic variable (pleasing and tallness) using direct rating. They defined three diagrams, one based on the mean value of rating and two others by adding and deducting the double value of standard deviation to mean point value in a way that all the membership grades are greater than *0* and smaller than the maximum scale rate. Consider the table 6; the information in this table shows the rating of '*warmth*' fuzzy concept for water by 40 people using a ten grade scale for temperature degrees between *0°* to *100°*. If we form the diagrams using the mentioned method we reach to diagrams of figure 11.a, due to a direct rating of this range of temperature degrees and figure 11.b for corresponding fuzzy sets diagrams.

Table 6
Frequency chart of warmth rating scale for a set of temperature degrees.

Rating scale	0°	20°	40°	50°	60°	80°	100°
0	40	1	0	0	0	0	35
1	0	2	0	0	0	0	5
2	0	20	0	0	0	5	0
3	0	10	0	0	0	10	0
4	0	5	0	0	0	20	0
5	0	2	2	0	0	5	0
6	0	0	11	0	0	0	0
7	0	0	16	0	4	0	0
8	0	0	10	5	10	0	0
9	0	0	1	15	16	0	0
10	0	0	0	20	10	0	0
Mean	0	2.55	6.93	9.38	8.80	3.63	0.13
Std. Dev.	0	1.04	0.92	0.70	0.94	0.87	0.33
Mean-2Stdv.	0	0.48	5.09	7.98	6.92	1.89	0
Mean+2Stdv.	0	4.62	8.76	10.78	10.68	5.36	0.79

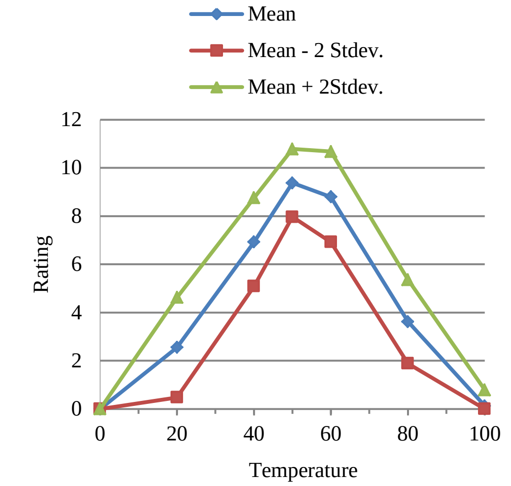


Fig. 11.a. Warm water rating diagrams using direct rating method.

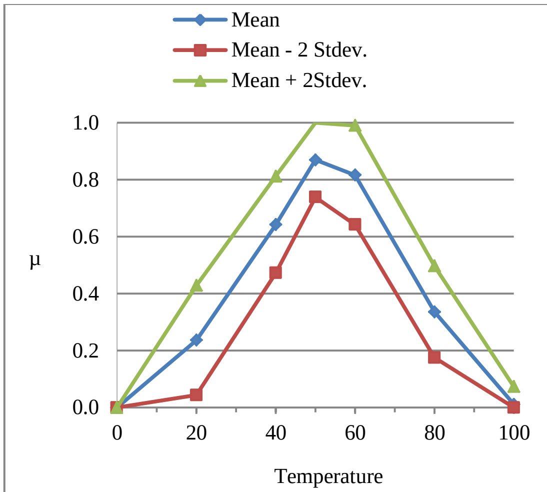


Fig. 11.b. Fuzzy sets of warm water using direct reting method.

6.2.3. The proposed method approach

By using the frequency chart data of table 6, we can summarize the proposed method parameters as shown in table 7.

Table 7
Proposed method parameters for a set of temperature degrees using the frequency chart of "warm water rating scale".

Method Parameters	0°	20°	40°	50°	60°	80°	100°
Mean of rating	0	2.55	6.93	9.38	8.80	3.63	0.13
Std. Dev.	0	1.04	0.92	0.70	0.94	0.87	0.33
A_L	0	20.49	18.81	15.98	16.92	15.70	4.14
A_R	0	16.51	19.47	11.52	16.08	19.30	15.86
$S=(A_L/A_R)$	0	1.24	0.97	1.39	1.05	0.81	0.26
σ_L	0	0.57	0.45	0.41	0.48	0.39	0.07
σ_R	0	0.46	0.47	0.30	0.46	0.48	0.27
LL (µ=0)	0	1.98	6.47	8.97	8.32	3.24	0.06
Mean (µ=1)	0	2.55	6.93	9.38	8.80	3.63	0.13
UL (µ=0)	0	3.01	7.39	9.67	9.26	4.10	0.39

We calculate the sub areas segregated on the frequency histogram by indicating the mean point of assessment on the rating scale axis (Fig. 12.a) and after determining the upper and lower limits we can form the triangular fuzzy number of each temperature degree that represents: "how that temperature degree is warm" (Fig. 12.b).

When a utility view of a fuzzy concept for different types of variables is considered, using this method is very appropriate, especially when we want to determine the fuzzy multi-attribute decision-making matrix (FMADM) weights; where each decision making factor is different and has its weight and impact factor. For example, imagine we want to form the decision-making matrix for evaluating several projects to different risk factors (cost, political and technical) where each risk factor impact or importance degree is needed to enter to the decision-making matrix as a triangular fuzzy number which represents how that factor is important.

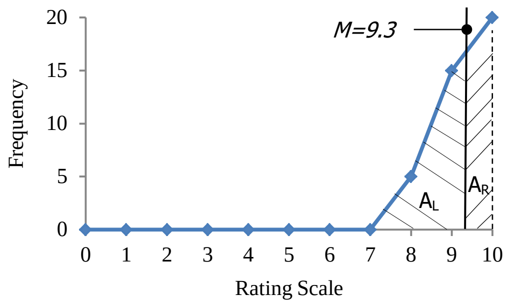


Fig. 12.a. Frequency histogram of 50° warmth rating scale.

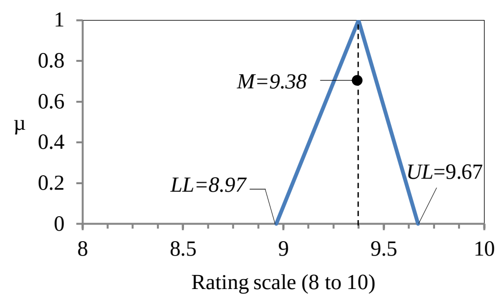


Fig. 12.b. TFN of "how 50°water is warm" using the proposed method.

Figure 13.a illustrates the diagrams of the calculated boundaries (mean, LL and UL) presented in table 7, for '*warmth'* fuzzy concept for a range of temperature degrees from 0° to 100° and figure 13.b is the scaled corresponding diagrams to [0,1]. In this case, we can determine the membership degree of each temperature degree in three states of most possible values (mean), highest possible values (Upper Limits) and lowest possible values (Lower Limits).

From the perspective of fuzzy logic, the space between the diagrams is the space arisen from vagueness and uncertainty. In the 100 percent certainty state, we only have one value for each variable, which is equal to the average value of ratings. In this case, the standard deviation of data will be zero (0), and these three diagrams coincide.

The fuzzy sets illustrated in figure 13.b may not be fuzzy numbers because as we mentioned in later sections, the fuzzy set must be convex and normal to be considered a fuzzy number as well. However, using the linear regression and finding the trend line is not part of the introduced method in this paper, it can be used as a solution for forming the triangular fuzzy number out of three sets of data (lower limit, mean and upper limit) produced by this method.

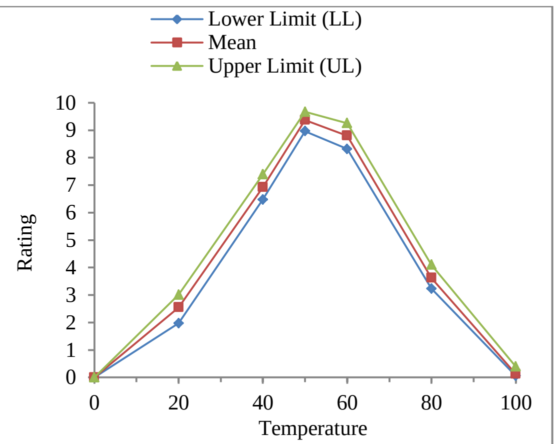


Fig. 13.a Warm water rating diagrams using the proposed method.

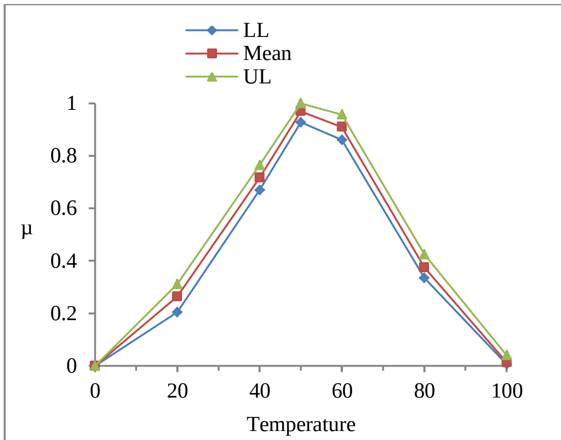


Fig. 13.b Fuzzy sets of warm water using proposed method.

Figure 14.a shows the triangular forms due to the linear regression of each three diagrams (LL, Mean, and UL). The equations of ultimate linear regression for all set of data have been shown in figure 14.b these equations result in the ultimate triangular shape of figure 14.c by scaling this shape to [0,1] we reach to a normal TFN (Fig.14.d.).

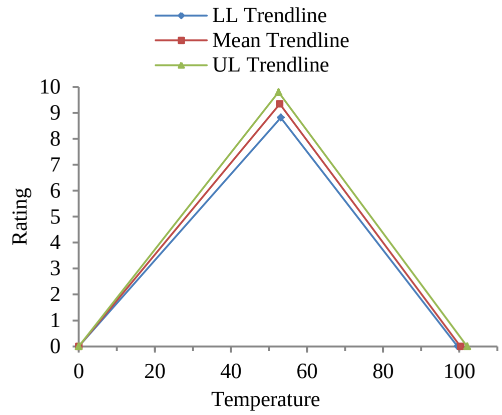


Fig. 14.a. Linear trendlines of warm water rating diagrams.

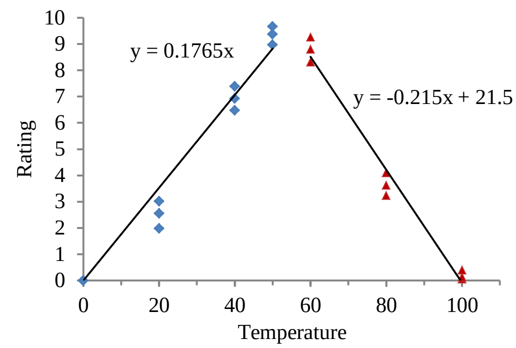


Fig. 14.b. Ultimate linear regression equations of LL, Mean and UL points.

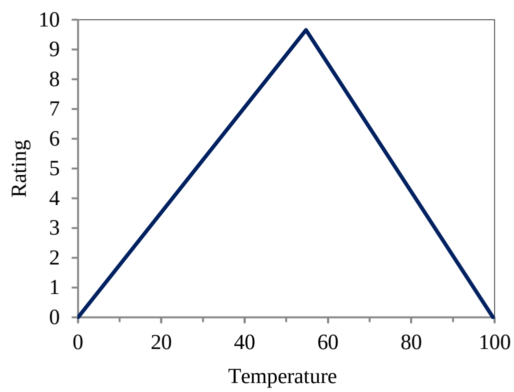


Fig. 14.c. Ultimate linear trendline of LL, Mean and UL points.

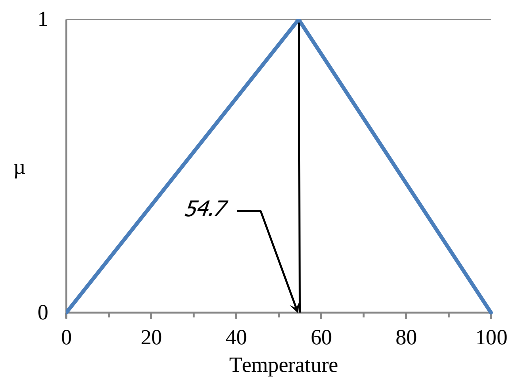


Fig. 14.d. TFN of warm water derived from ultimate linear trendline.

We can show the triangular fuzzy number of figure 14.d in equation 17. The possible value of this TFN with membership degree *1* is *54.7°*. It means the water with this degree can be considered warm water with maximum certainty.

$$\text{Triangular}\ (x; 0,54.7,100) \begin{cases} 0 & x \le 0 \\ \frac{(x-54.7)}{54.7} & 0 \le x \le 54.7 \\ \frac{(100-x)}{45.3} & 54.7 \le x \le 100 \\ 0 & 100 \le x \end{cases} \tag{17}$$

The main advantages and properties of this method can be listed as followings:

- It is simple, quick and functional.
- It makes the space between rating scale grades meaningful.
- This method produces exclusive TFNs for each set of data even with same mean values, different distributions of frequency chart result in different TFN shapes.
- This method tries to emphasize the uncertainties hidden in subjective perceptions and direct rating method, which is the main idea of fuzzy logic.

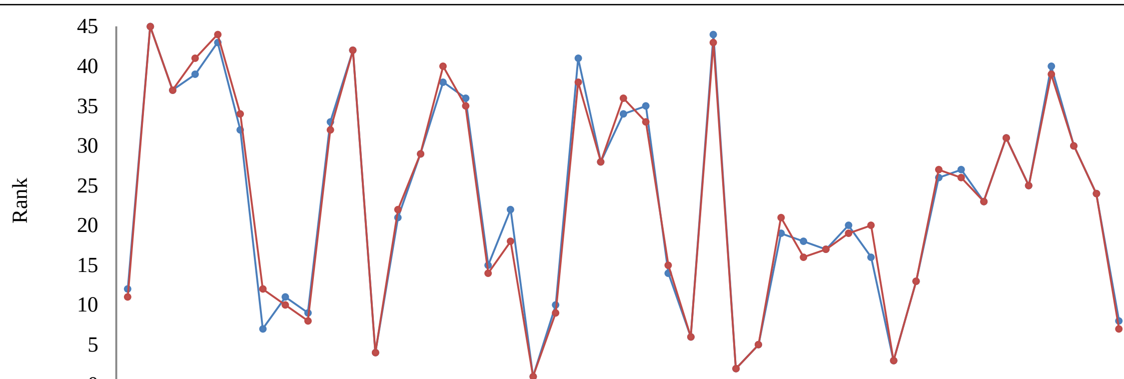
7. Verification of the proposed method

The method proposed in this paper was used in a study which has been formed and carried out by the author based on a framework to apply fuzzy concepts and logic in bridge management field [30]. In that research one of the defined problems was evaluating, ranking, and assign fuzzy weights to the parameters which were effective for prioritizing the urban roadway bridges for maintenance operations. In that study, 45 parameters were identified under four main categories: destruction, bridge damage consequences, cost and facilities and strategic factors. We had to assign 45 fuzzy triangular numbers to these 45 parameters to use them as fuzzy weights in fuzzy decision-making matrix and rank them using fuzzy ranking methods. So after identifying the parameters, their degree of importance and effectiveness in bridge maintenance operations were surveyed through a closed questionnaire in a 10 distance elaborated scale by 80 bridge experts of four main groups of contractor, consultant, researcher, and employer. Numbers from 0 to 10 were assigned to 11 linguistic variables that defined the importance degree of parameters in a range from unimportant to a special important degree.

The verification was performed in two aspects. First, in ranking the parameters and second, in selecting the most important parameters for further process.

7.1. Ranking the parameters

After collecting the completed questionnaires, their data were analyzed by using classic statistical methods and parameters were ranked by using the Friedman test [31] then the result was compared with the produced TFNs' fuzzy ranking output. In this study, the Mabuchi [32] algorithm was used for ranking the fuzzy numbers. This method proposes a ranking method by using multiple levels of α-cut which will have the weights role. Figure 15 shows two diagrams represent this comparison. The blue diagram illustrates the ranking result using Mabuchi method for TFNs constructed by using the proposed method and the red diagram indicates the ranking based on the classic method of the Friedman test. In figure 15, P1 to P45 are parameters' row numbers in the questionnaire.



	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15
Fuzzy	12	45	37	39	43	32	7	11	9	33	42	4	21	29	38
Friedman	11	45	37	41	44	34	12	10	8	32	42	4	22	29	40

	P16	P17	P18	P19	P20	P21	P22	P23	P24	P25	P26	P27	P28	P29	P30
Fuzzy	36	15	22	1	10	41	28	34	35	14	6	44	2	5	19
Friedman	35	14	18	1	9	38	28	36	33	15	6	43	2	5	21

	P31	P32	P33	P34	P35	P36	P37	P38	P39	P40	P41	P42	P43	P44	P45
Fuzzy	18	17	20	16	3	13	26	27	23	31	25	40	30	24	8
Friedman	16	17	19	20	3	13	27	26	23	31	25	39	30	24	7

Fig. 15. Parameters ranking diagrams based on fuzzy approach and Friedman test method.

7.2. Selection the most effective parameters

In a non-fuzzy study [33], selecting the most important parameters which affect the bridge priority for maintenance operations was based on this fact that the number of parameters should not be limited as much as to raise the prioritizing error. Also, they should not be such extended that encounter analysis process with complexity. So after performing the ranking using the Friedman test rank value, on the corresponding diagram that shows the rank value of priority numbers (Fig. 16), 24 parameters before the point that a big fracture appears in the diagram, were selected as most important parameters.

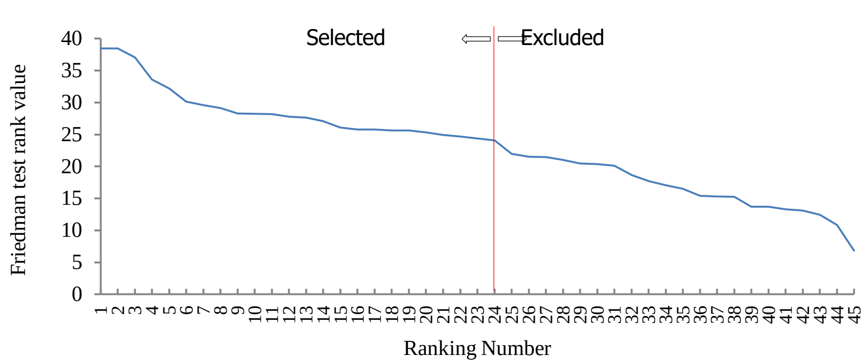


Fig. 16. Selection of the most effective parameters ranked by Friedman test.

In the fuzzy approach, for selecting the most effective parameters, those that their minimum fuzzy desirability of 50% with $\alpha = 0.5$ is below the average index of the importance degree are excluded [28]. To avoid the complexity of the drawing, a schematic diagram of the determination of the 50% fuzzy desirability has been illustrated for 4 parameters in figure 17. In this diagram the minimum fuzzy desirability of 50 % for P2, P27 and P24 are below the average importance degree (5), so they would be excluded. Thirty-two parameters out of 45 parameters (4 more parameters than the non-fuzzy approach) were selected in this way for further process. This case shows the fuzzy uncertainties involvement in determining the parameters prioritization. Wider range and stronger uncertainties involved in the fuzzy ranking process for selecting the most effective parameters are such cases that cannot be seen in classic statistical methods such as Friedman test.

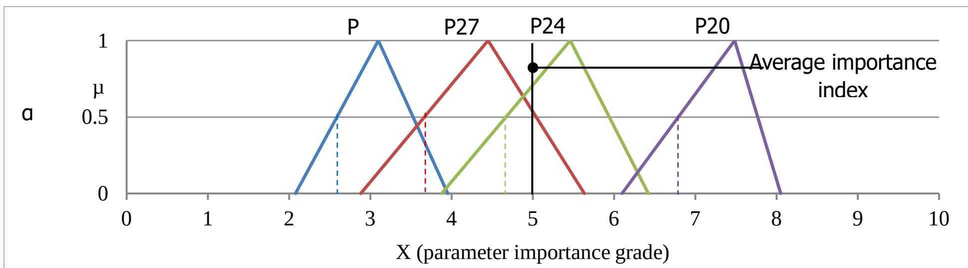


Fig. 17. Using minimum fuzzy desirability of 50% for selecting the most important parameters(TFNs have been produced using the proposed method)

8. Conclusion

Using the fuzzy logic is a solution to overcome the limitations of decision making in an uncertain environment or analysis, judgment and evaluating values or concepts where there is a lack of transparency or imperfect information. In other words, fuzzy logic covers a wider area of judgments includes the vagueness. In this paper, a simple algorithm for constructing the triangular membership function was presented based on a direct rating method using the frequency chart data of a rated variable on a numerical graded scale. These grades can be a conversion of oral judgment or qualitative assessment of a fuzzy parameter or concept. In the proposed method we used statistical values of average and standard deviation to form the boundaries of TFN, in a way that represents the uncertainty of parameter assessment, which is evident in the distribution of frequency chart. In the described algorithm only a symmetrical distribution of frequency diagram about the average index leads to an isosceles triangle fuzzy number and when there is a 100 percent certainty about the parameter assessment we can see a fuzzy singleton number without any boundaries. Using this method in cases that the evaluated parameters are the importance degree or weight factors of a multi-criteria decision-making matrix can reflect the assessment of uncertainties in a reasonable way.

In the last section of this article, we verified the proposed method with non-fuzzy classic methods in a practical case study by comparing the fuzzy ranking of fuzzy triangular numbers of 45 parameters, constructed using the proposed method, with the Friedman test rank values. This verification justifies the partial differences in output results due to considering the uncertainties of qualitative assessments.

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